On the gravitational interaction of plane symmetric clouds of null dust[a)]

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Plane symmetric solutions of the Einstein field equations are considered, solutions that represent the collision of oppositely moving clouds of initially unidentified massless particles—clouds of "null dust." In terms of specific examples it is shown that the corresponding Cauchy problem remains ambiguous even when the Riemann tensor is free of any kind of discontinuity. By solving the corresponding Einstein–Klein–Gordon and Einstein–Maxwell–Weyl field equations the ambiguity is resolved and space-time models are constructed representing, among others, various types of collisions between a pair of scalar wave pulses as well as of a pulse of electromagnetic waves with a cloud of neutrinos.

I. INTRODUCTION

Two classes of solutions of the Einstein field equations obtained by Chandrasekhar and Xanthopoulos[1,2] (CX) recently, solutions that represent the collision of "null dust clouds" (beams of massless particles), have given rise to a pair of interrelated issues. The first concerns the uniqueness of the outcome of a collision of the above kind. It arose from the fact that the two CX solutions referred to above are geometrically identical before the collision but differ from each other in the region where the beams interact. The second question raised in the context of the CX solutions is whether it is classically possible for massless particles to get transformed into massive ones by interacting gravitationally. This question arose from the fact that in the first of the above solutions a perfect fluid with a p (pressure) $= \varepsilon$ (energy density) equation of state appears in the region where the null dust clouds interact.

Regarding the question of uniqueness, Taub's[3,4] recent analysis of space-time models representing the collision of null dust shells whose leading fronts are occupied by impulsive (Dirac delta) distributions of null dust or impulsive gravitational waves has revealed the fact that the corresponding Cauchy problem is ambiguous indeed. Taub also determined the conditions that must be imposed on the stress-energy tensor in the region of interaction in order for the initial value problem to become unambiguous.

Feinstein, MacCallum, and Senovilla[5] (FMS), however, have recently argued that the above problem remains indeterminate even after Taub's points are taken into account. According to the above authors, the remaining ambiguities are removed "by a more complete treatment of the matter content." By the latter they essentially mean that, besides the gravitational field equations, one has to consider the equations governing the evolution of the matter fields present and impose the boundary conditions that are appropriate to the latter fields, too. As for the second issue raised in the context of the CX solutions, their approach lead FMS to the belief that "the transformation of a general null dust into other forms of matter is unrealizable in classical general relativity."

We tend to believe that, as far as the problem of uniqueness is concerned, the FMS approach does not improve Taub's analysis. As shown by Taub the conditions that must be imposed on the stress-energy tensor in the region of interaction in order to render the Cauchy problem unambiguous are equivalent to specifying the manner in which the incident null currents interact. When, on the other hand, one follows the FMS instructions of first specifying the matter fields involved in the problem, one does no more than to provide the conditions required by Taub by a different means.

However, the FMS approach becomes imperative when one decides to interpret the null dust currents appearing in Taub's picture in terms of known matter fields and their interactions. The matter fields picture is more satisfying to the physicist seeking models for "real life" processes. In short, the difference between the Taub and FMS pictures reflects no more, but no less, than the well-known "problem" of general relativity, the "problem" of the right-hand side of Einstein's equations.[6]

In any case, our purpose in the present paper is not to deal with the issues described above in all of their generality. We concentrate, instead, on providing more test cases by constructing new classes of solutions representing the collision of null dust clouds. Specifically, we consider solutions of the Einstein field equations

$$R_{ab} - \tfrac{1}{2}g_{ab}R = -(8\pi G/c^2)T_{ab}, \quad (1.1)$$

where a,b range from 0–3, when subjected to the following conditions.

(i) The space-time manifold is plane symmetric. This means that a set $(x^0,x^1,x^2,x^3) \equiv (u,v,x,y)$ of coordinates exists in terms of which the line element can be written in the form

$$ds^2 = g_{ab}dx^a\,dx^b = 2e^{-M}du\,dv - e^{-U}(dx^2 + dy^2), \quad (1.2)$$

where M,U are functions of u and v, only.

[a)] Dedicated to the memory of Professor Basilis C. Xanthopoulos whose untimely end deprived us of a precious friend and collaborator.

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(ii) The Ricci tensor R_{ab} is such that

$$R_{xx} = R_{yy} = 0 \tag{1.3}$$

everywhere, and

(iii) in an open domain of the (u,v) plane, referred to hereafter as the region of interaction,

$$U_{,u} U_{,v} \equiv \frac{\partial U}{\partial u} \frac{\partial U}{\partial v} \neq 0 \tag{1.4}$$

and either

$$R_{uu} R_{vv} = R_{uv}{}^2 \neq 0 \tag{1.5}$$

or

$$R_{uv} = 0. \tag{1.6}$$

According to whether they satisfy condition (1.5) or (1.6), the solutions under consideration will be referred to in the following as being of type A or B, respectively.

In this direction, a new six-parameter class of type B metrics is obtained which represents the collision of a pair of plane symmetric clouds of null dust made of photons and neutrinos, respectively. The solution of the corresponding Einstein–Maxwell–Weyl equations is presented in Sec. V. In the same section, the pertinence of the above solution to the issue of the ambiguous evolution of colliding null dust clouds is established on the basis of the following result. A three-parameter subset of our type B metrics is identical, in the precollision region, to a class of type A models that were recently obtained by Ferrari and Ibañez.[7]

In deriving the above solutions we take the opportunity to reexamine certain aspects of Taub's's[3] recent study of the whole class of metrics satisfying Eqs. (1.1)–(1.6). We are thereby led to the following interesting results. (i) Type A models can be constructed directly, i.e., without any integrations, from solutions of the Einstein vacuum equations corresponding to the metric

$$ds^2 = 2e^{-L} du\, dv - e^{-U}(e^V dx^2 + e^{-V} dy^2), \tag{1.7}$$

where V is also a function of u and v, only. (ii) By not restricting one's choice of gauge to Taub's $\exp(-U) = 1 - Au - Bv$, with A,B constants, one can obtain solutions in which the Riemann curvature tensor is continuous across the separation hypersurfaces $u = 0$ and $v = 0$. Beyond its usefulness in constructing new exact models, this result dissolves the impression that may arise from the CX models and Taub solutions that the "ambiguity" in the evolution of colliding null dust clouds is intimately connected with the presence of impulsive (Dirac delta shaped) gravitational waves or null dust pulses on the null hypersurfaces $u = 0$ and $v = 0$.

The above results are established in Secs. II and III, respectively. Section IV, on the other hand, is devoted to a short analysis of the Ferrari–Ibañez (FI) solutions mentioned above, following their derivation by the algorithm presented in Sec. II. The purpose served by this discussion goes beyond the necessity of establishing the effectiveness of the above algorithm. First of all, it is thereby made convenient to compare, from the point of view of geometry as well as matter field content, the type B solutions of Sec. V to their type A counterparts. Moreover, it allows us to give an explicit example of the superiority of the FMS or matter fields picture, as compared to the null currents one, in better understanding the process of collision. We show, in particular, that the production of an $\varepsilon = p$ perfect fluid in some of the FI models and, by analogy, in the CX solutions is the result of a special relation between the phases of the incoming waves.

II. SOLUTIONS OF THE EINSTEIN EQUATIONS

The nonvanishing compoents of the Ricci tensor corresponding to the line element (1.1) are given by

$$R_{uu} = -\tfrac{1}{2}(2U_{,uu} - U_{,u}^2 + 2U_{,u} M_{,u}), \tag{2.1a}$$

$$R_{vv} = -\tfrac{1}{2}(2U_{,vv} - U_{,v}^2 + 2U_{,v} M_{,v}), \tag{2.1b}$$

$$R_{uv} = R_{vu} = -\tfrac{1}{2}(2M_{,uv} + 2U_{,uv} - U_{,u} U_{,v}), \tag{2.1c}$$

$$R_{xx} = R_{yy} = e^{M-U}(U_{,uv} - U_{,u} U_{,v}), \tag{2.1d}$$

where $U_{,uu} \equiv \partial^2 U/\partial u^2, U_{,uv} \equiv \partial^2 U/\partial_u \partial_v$, etc.

Therefore, condition (1.3) becomes

$$(e^{-U})_{,uv} = -e^{-U}(U_{,uv} - U_{,u} U_{,v}) = 0. \tag{2.2}$$

Assuming that in the open region of the (u,v) plane under consideration condition (1.4) holds, Eq. (2.2) can be immediately solved to give

$$e^{-U} = \alpha(u) + \beta(v), \tag{2.3}$$

where α,β are arbitrary nonconstant functions of their respective arguments.

Thus in order to find the second function M, which together with U determines the geometry of the space-time models under study, it remains to specify the stress-energy tensor T_{ab} and integrate the Einstein equations corresponding to R_{uu}, R_{vv}, and R_{uv}. At this point we restrict our considerations to solutions belonging to one of the following two types.

A. Solutions of type A

These are solutions that satisfy condition (1.5). As shown by Taub,[3] the corresponding Ricci tensor takes the form,

$$R_{ab} = -\varphi_{,a}\varphi_{,b}, \tag{2.4}$$

where $\varphi = \varphi(u,v)$ satisfies the equation

$$2\varphi_{,uv} - U_{,u}\varphi_{,v} - U_{,v}\varphi_{,u} = 0. \tag{2.5}$$

Thus, in the present case, the source term of the Einstein equations is due to a massless scalar field φ and can be written as

$$T_{ab} = \varphi_{,a}\varphi_{,b} - \tfrac{1}{2} g_{ab}\varphi_{;c}\varphi^{;c}, \tag{2.6}$$

where the semicolon denotes covariant differentiation and the units were chosen such that $8\pi G = c = 1$.

It now follows from Eqs. (2.1a)–(2.1c) and (2.4) that the Einstein equations remaining to be integrated read

$$2U_{,uu} - U_{,u}{}^2 + 2U_{,u} M_{,u} = 2\varphi_{,u}{}^2, \tag{2.7a}$$

$$2U_{,vv} - U_{,v}{}^2 + 2U_{,v} M_{,v} = 2\varphi_{,v}{}^2 \tag{2.7b}$$

$$2M_{,uv} + U_{,uv} = 2\varphi_{,u}\varphi_{,v}. \tag{2.7c}$$

In fact, the last of Eqs. (2.7) is no more than the integrability condition of the first two. Therefore, given U and a solution φ of Eq. (2.5), M can be determined by line integration.

The type A plane symmetric solutions of Einstein's equations were first studied as a class by Tabensky and Taub[8] who were able to construct the Riemann function for the corresponding initial value problem using the gauge $\exp(-U) = (u+v)/2$. From the practical point of view, however, it is worthwhile to observe that all the solutions of the system of Eqs. (2.3), (2.5), and (2.7) can be immediately obtained from the solutions of the Einstein vacuum equations corresponding to the metric (1.7).

To prove this claim, let us first note that writing M in the form

$$M = N - \Omega, \tag{2.8}$$

with

$$N = \tfrac{3}{2}U - \ln|2U_{,u}U_{,v}|, \tag{2.9}$$

and substituting it into Eqs. (2.7a)–(2.7c), we obtain

$$\Omega_{,u} = -\varphi_{,u}^2/U_{,u},$$

$$\Omega_{,v} = -\varphi_{,v}^2/U_{,v}, \text{ and } \Omega_{,uv} = -\varphi_{,u}\varphi_{,v}, \tag{2.10}$$

respectively.

Consider, next, the Einstein equations corresponding to the line element (1.7) with $T_{ab} = 0$. When L is written in the form

$$L = N - \Sigma, \tag{2.11}$$

they give[9]

$$U_{,uv} - U_{,u}U_{,v} = 0, \tag{2.12}$$

$$2V_{,uv} - U_{,u}V_{,v} - U_{,v}V_{,u} = 0, \tag{2.13}$$

and

$$\Sigma_{,u} = -\frac{V_{,u}^{\ 2}}{2U_{,u}},$$

$$\Sigma_{,v} = -\frac{V_{,v}^{\ 2}}{2U_{,v}}, \text{ and } \Sigma_{,uv} = -\frac{V_{,u}V_{,v}}{2}. \tag{2.14}$$

Comparing Eqs. (2.12)–(2.14) with (2.2), (2.5), and (2.10), respectively, leads to the following conclusion. Given a solution (U,V,Σ) of the former, one obtains a solution of the latter set of equations by taking $\varphi = \lambda V/\sqrt{2}$ and $\Omega = \lambda^2\Sigma + \Omega_0$, where λ,Ω_0 are arbitrary constants. In fact, Ω_0 can be neglected and the above conclusion can be given in the form of the following.

Theorem: Let (L,U,V) be a solution of the Einstein vacuum equations corresponding to the metric (1.7) and (λ,μ) a pair of arbitrary constants satisfying the condition $\lambda^2 - \mu^2 = 1$. Then $(M,U) = (\lambda^2 L - \mu^2 N, U)$, with N given by (2.9), is a solution of the Einstein equations corresponding to the metric (1.2) and a nonvanishing stress-energy tensor due to a massless scalar field $\varphi = \lambda V/\sqrt{2}$.

B. Solutions of type B

Suppose now that condition (1.6), instead of (1.5), holds. Then we can combine Eqs. (2.1c) and (2.2) so as to write the above condition in the form

$$2M_{,uv} + U_{,uv} = 0. \tag{2.15}$$

If we now introduce the function $\Lambda(u,v)$ via the relation

$$M = -\tfrac{1}{2}U - \Lambda, \tag{2.16}$$

we find, using Eq. (2.15), that Λ satisfies the wave equation

$$\Lambda_{,uv} = 0. \tag{2.17}$$

Therefore,

$$\Lambda = A(u) + B(v), \tag{2.18}$$

where A,B are arbitrary functions.

Returning to Eq. (2.3), we find that in the present case the Ricci tensor takes the form

$$R_{ab} = \frac{\alpha'' - \alpha'A'}{\alpha+\beta}k_a k_b + \frac{\beta'' - \beta'B'}{\alpha+\beta}l_a l_b, \tag{2.19}$$

where the prime denotes ordinary differentiation and (k_a,l_a) is a pair of null vectors defined by

$$k_a = \delta_a^0, \quad l_a = \delta_a^1. \tag{2.20}$$

Equivalently, the stress-energy tensor corresponding to type B solutions can be written in the form

$$T_{ab} = T_{ab}^L + T_{ab}^R, \tag{2.21}$$

where

$$T_{ab}^L = \varepsilon^L k_a k_b \equiv [(\alpha'A' - \alpha'')/(\alpha+\beta)]k_a k_b, \tag{2.22a}$$

$$T_{ab}^R = \varepsilon^R l_a l_b \equiv [(\beta'B' - \beta'')/(\alpha+\beta)]l_a l_b. \tag{2.22b}$$

If one combines Eqs. (2.21) and (2.22) with the easily deduced fact that

$$T_{ab}^{L\ ;b} = 0 = T_{ab}^{R\ ;b}, \tag{2.23}$$

one is led to the following physical picture. Assuming that both ε^L and ε^R are positive, the source of type B models consists of a pair of oppositely moving null dust clouds of energy density ε^L and ε^R, respectively, each of which is separately conserved.

III. EXTENSION OF THE SOLUTIONS

Let us suppose that a solution of Einstein's equations of the kind considered in the last section is given in the region of interaction, which it will be now assumed to consist of that open region of the (u,v) plane that has the curve

$$C = \{(u,v): 0 \leqslant u < \xi_1, v = 0\} \cup \{(u,v): u = 0, 0 \leqslant v \leqslant \xi_2\},$$

where $\xi_1,\xi_2 \in \mathbb{R}^+$, as its past boundary. Let the above region be denoted by IV and let region III $= \{(u,v): 0 < u < \xi_1, v < 0\}$, region II $= \{(u,v): u < 0, 0 < v < \xi_2\}$ and region I $= \{(u,v): u < 0, v < 0\}$.

If the functions determining the given solution are smooth inside region IV and remain bounded (but different from zero) as one approaches the curve C, the solution can be easily extended across C using the well-known algorithm of Khan and Penrose.[10] The latter consists of letting

$$f(u,v) \to \bar{f}(u,v) \equiv f(uH(u), vH(v)), \tag{3.1}$$

where f is any of the functions appearing in the solution given in region IV and $H(x)$ the unit step function of Heaviside. Since by this manner of extension the metric coefficients, as well as the source fields, become functions of only v (u) in region II (III) and constants in region I, one is immediately led the following interpretation of the extended solution. A pair of plane waves having their leading wave fronts along the $u=0$ and $v=0$ characteristics, respectively, are incident from opposite sides of region I which is flat and collide at the point $(u,v)=(0,0)$ where regions I–IV meet.

The above interpretation presupposes, however, that the Einstein as well as the matter field equations are satisfied not only in regions I–III, but on the separation hypersurfaces $u=0$ and $v=0$, as well, after substitution (3.1) has been applied. That the above conditions are satisfied in the case under consideration is easy to verify, provided one now understands the Einstein equations as holding in the sense of distributions. Specifically, in the case of type A models one must replace Eq. (2.4) by

$$\begin{aligned} R_{ab} &= -\bar{\varphi}_{,a}\bar{\varphi}_{,b} - U_{,u}(0,vH(v))\delta(u)k_a k_b \\ &\quad - U_{,v}(uH(u),0)\delta(v)l_a l_b \\ &= -[\varphi^2_{,u}(u,vH(v))H(u) + U_{,u}(0,vH(v))\delta(u)]k_a k_b \\ &\quad - [\varphi^2_{,v}(uH(u),v)H(v) + U_{,v}(uH(u),0)\delta(v)]l_a l_b \\ &\quad - \varphi_{,u}\varphi_{,v}H(u)H(v)(k_a l_b + k_b l_a), \end{aligned} \tag{3.2}$$

where δ denotes the Dirac distribution, or delta function.

By similar considerations one is led to the conclusion that, in the case of type B models, Eq. (2.22) must be replaced, upon extension, by

$$\begin{aligned} R_{ab} &= (\alpha+\beta)^{-1}[(\alpha'' - \alpha' A')H(u) + \alpha'\delta(u)]k_a k_b \\ &\quad + (\alpha+\beta)^{-1}[(\beta'' - \beta' B')H(v) + \beta'\delta(v)]l_a l_b, \end{aligned} \tag{3.3}$$

where the argument of α, A is now $uH(u)$ and that of β, B $vH(v)$.

IV. COLLIDING SCALAR WAVES

An example of type A solutions which can be constructed using the results of the last two section is obtained along the following lines.

We first choose the functions α and β figuring in Eq. (2.3) to be

$$\alpha = \tfrac{1}{2} - u^{2n}, \quad \beta = \tfrac{1}{2} - v^{2m}, \tag{4.1}$$

respectively, where

$$n = \tfrac{1}{2} \text{ or } n \geqslant 1 \text{ and } m = \tfrac{1}{2} \text{ or } m \geqslant 1. \tag{4.2}$$

For convenience we also introduce the pairs of functions (t,z) and (η,μ) defined by

$$t = \alpha + \beta = e^{-U} = 1 - u^{2n} - v^{2m}, \tag{4.3a}$$

$$z = -\alpha + \beta = u^{2n} - v^{2m}, \tag{4.3b}$$

and

$$\eta = u^n(1-v^{2m})^{1/2} + v^m(1-u^{2n})^{1/2}, \tag{4.4a}$$

$$\mu = u^n(1-v^{2m})^{1/2} - v^m(1-u^{2n})^{1/2}, \tag{4.4b}$$

respectively. Each of the above pairs, which are related by

$$t = (\Delta\delta)^{1/2}, \quad \Delta \equiv 1 - \eta^2, \quad \delta \equiv 1 - \mu^2, \tag{4.5a}$$

$$z = \eta\mu, \tag{4.5b}$$

can replace (u,v) in the role of coordinates in region IV. In such a case, the metric takes the form

$$ds^2 = t^{-1/2}e^{\Omega}(dt^2 - dz^2) - t(dx^2 + dy^2) \tag{4.6}$$

or

$$ds^2 = \frac{\eta^2 - \mu^2}{(\Delta\delta)^{1/4}} e^{\Omega}\left(\frac{d\eta^2}{\Delta} - \frac{d\mu^2}{\delta}\right) - (\Delta\delta)^{1/2}(dx^2 + dy^2), \tag{4.7}$$

respectively. Let us now choose the vacuum solution figuring in the theorem of Sec. II to be the one given in Ref. 11 and $\lambda = \sqrt{2}$. Then a three-parameter family of type A solutions is obtained which is defined by Eq. (4.3) and the following expressions:

$$\begin{aligned} \varphi &= V \equiv a \ln(1-\eta^2)(1-\mu^2) + \delta_1 \ln\left(\frac{1-\eta}{1+\eta}\right) \\ &\quad + \delta_2 \ln\left(\frac{1-\mu}{1+\mu}\right), \end{aligned} \tag{4.8}$$

$$\begin{aligned} e^{-M} &= (1-\eta)^{b_1}(1+\eta)^{c_1}(1-\mu)^{b_2}(1+\mu)^{c_2} \\ &\quad \times S^{-2\delta_+^2} W^{-2\delta_-^2}, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} &S = (1-v^{2m})^{1/2}, \qquad W = (1-u^{2n})^{1/2}, \\ &\delta_+ = \delta_1 + \delta_2, \qquad \delta_- = \delta_1 - \delta_2, \\ &b_A = 2(a+\delta_A)^2 - \tfrac{1}{4}, \quad c_A = 2(a-\delta_A)^2 - \tfrac{1}{4}, \end{aligned} \tag{4.10}$$

and a, δ_A $(A = 1,2)$ are real constants, the last two of which are related to n and m of Eq. (4.2) by

$$\delta_+^2 = 1 - 1/2n, \quad \delta_-^2 = 1 - 1/2m, \tag{4.11}$$

As it stands, the solution just obtained holds only in region IV. The results of Sec. III, however, allow us to claim that, in fact, the validity of the above solution extends to all regions I–IV of Fig. 1, including the characteristics $u=0$ and $v=0$. Specifically, the metric coefficients are given by Eqs. (4.3), and (4.9), respectively, where the pair (u,v) is now understood to stand for $(uH(u),vH(v))$. The same holds for Eq. (4.8) which specifies the massless Klein–Gordon or scalar field that determines the stress-energy tensor of the solution.

It can be easily shown that the above solution is a member of the FI four-parameter class of metrics given in Ref. 7. In fact, it is that three-parameter subfamily of metrics which are described as "stiff matter solutions" in the above reference. The following considerations, however, show that such a term does not do justice to the physically rich class of models defined by Eqs. (4.3), (4.8), and (4.9).

To begin with, a massless scalar field is not energetically equivalent to a $p = \varepsilon$ perfect fluid always. Writing the stress-energy tensor given by Eq. (2.6) in the form

$$T_{ab} = \varphi_{,a}\varphi_{,b} - \varepsilon g_{ab}, \tag{4.12}$$

where

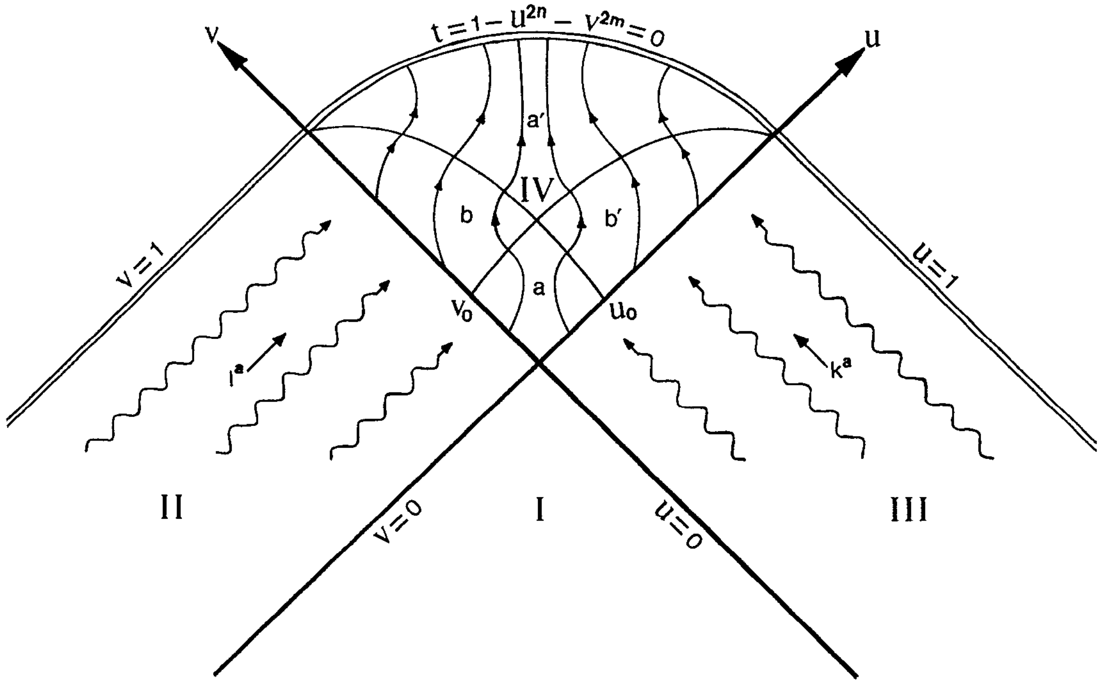

FIG. 1. This figure is representative of all the models of colliding plane waves discussed in the text. The particular case illustrated corresponds to a pair of scalar wave pulses, with propagation vectors k^a and l^a, respectively, incident from opposite sides of region I, which is flat. Region IV, the space-time region where the waves interact gravitationally, may be thought of as being occupied by a matter distribution. Its fluid lines are depicted by the directed curves of region IV. In subregions a and a', the fluid equivalent to the interacting waves is "stiff matter," i.e., a perfect fluid with pressure equal to energy density. In subregions b and b' it becomes a nonconducting fluid whose principal stresses are equal to (energy density) $\times(-1,-1,1)$. In all cases a physical singularity develops along the curve $t=1-u^{2n}-v^{2m}$ topping region IV.

$$\varepsilon := \tfrac{1}{2}\varphi_{;a}\varphi^{;a}, \tag{4.13}$$

one realizes that, in general, the following possibilities arise.[8]

(i) $\varepsilon>0$: In this case the vector $\varphi_{,a}$ is timelike and the stress-energy tensor (4.12) takes the form corresponding to a perfect fluid with four-velocity proportional to $\varphi_{,a}$ and equation of state (energy density = pressure = ε).

(ii) $\varepsilon<0$: In this case $\varphi_{,a}$ is space-like and the scalar field φ becomes energetically equivalent to a fluid with (energy density) $=|\varepsilon|$ and (principal stresses) $=|\varepsilon|(-1,-1,1)$. In other words the direction $\varphi^{;a}$ is characterized by a pressure and the directions transverse to it by a tension, both equal to the energy density of fluid.

(iii) $\varepsilon=0$: In this case $\varphi_{,a}$ is a null vector and T_{ab} takes the form associated with what we have been calling a null dust cloud propagating in the direction defined by $\varphi^{;a}$.

In terms of Fig. 1, the last of above possibilities describes the situation in regions II or III, when the latter are not empty. In region IV, on the other hand,

$$\begin{aligned}\varepsilon &= -e^{M}R_{uv}\\ &= \frac{4nme^{M}\tan\chi\tan\omega}{t^{2}(\sin\chi)^{1/n}(\sin\omega)^{1/m}}\\ &\quad\cdot(a\sin 2\chi+\delta_{+}\cos\chi\cos\omega+\delta_{-}\sin\chi\sin\omega)\\ &\quad\cdot(a\sin 2\omega+\delta_{-}\cos\chi\cos\omega+\delta_{+}\sin\chi\sin\omega),\end{aligned} \tag{4.14}$$

where

$$\sin\chi=u^{n},\quad \sin\omega=v^{m}. \tag{4.15}$$

The consequences of the above observations are best illustrated by considering the following representative examples of the class of models under study.

Case A1—$\delta_A=0(n=m=1/2)$: When a is the only nonvanishing parameter, Eq. (4.8) yields the expression

$$\begin{aligned}\delta &= a\ln(1-\eta^{2})(1-\mu^{2})\\ &= a\ln[1-uH(u)-vH(v)],\end{aligned} \tag{4.16}$$

for the scalar field source. Using Eq. (3.2), on the other hand, we find that

$$\begin{aligned}R_{uu} &= 4a^{2}[1-u-vH(v)]^{-2}H(u)\\ &\quad-[1-vH(v)]^{-1}\delta(u),\end{aligned} \tag{4.17a}$$

$$\begin{aligned}R_{vv} &= -4a^{2}[1-u-H(u)-v]^{-2}H(v)\\ &\quad-[1-uH(u)]^{-1}\delta(v),\end{aligned} \tag{4.17b}$$

$$R_{uv}=R_{vu}=-4a^{2}(1-u-v)^{-2}H(u)H(v). \tag{4.17c}$$

Equations (4.17a) and (4.17b) show that (a) the incoming scalar plane waves are energetically equivalent to a pair of null dust clouds with a shock front, and (b) the incident waves are accompanied by impulsive shells of null dust. Equation (4.17c), on the other hand, combined with the considerations following Eq. (4.12), implies that, in the region of interaction, the scalar waves superpose so to become equivalent to a perfect fluid for which

$$p=\varepsilon=4a^2(1-u-v)^{-(\delta a^2+3)/2}=4a^2t^{-(\delta a^2+3)/2}. \quad (4.18)$$

The last equation shows explicitly that the $\delta_A=0$ models develop a physical singularity along the $t=0$ hypersurface. In fact, this behavior is representative of all the colliding wave models considered in this paper.

Case A2—$a=\delta_1=0$: In this case

$$\varphi=\delta\ln(1-\mu)/(1+\mu),\quad \delta\equiv\delta_2 \quad (4.19)$$

and

$$R_{uu}(v<0)=-4n^2\delta^2(1-u^{2n})^{-2}u^{2(n-1)}H(u), \quad (4.20a)$$

$$R_{vv}(u<0)=-4n^2\delta^2(1-v^{2n})^{-2}v^{2(n-1)}H(v), \quad (4.20b)$$

$$R_{uv}=-e^{-M}\varepsilon=\frac{4n^2\delta^2(\sin\chi\sin\omega)^{2\delta^2-1}\cos^2(\chi+\omega)}{t^2\cos\chi\cos\omega}\times H(u)H(v), \quad (4.20c)$$

where

$$n=m=1/2(1-\delta^2)\geqslant 1. \quad (4.21)$$

The first two of Eqs. (4.20) show that the $a=\delta_1=0$ subclass of models represent the collision of a pair of similar pulses of scalar waves none of which is accompanied by impulsive components. According to the last of the above equations, on the other hand, $\varepsilon<0$. Thus the fluid equivalent to the energy source of region IV is not of the "stiff matter" kind in this case, but of the type described as possibility (ii) after Eq. (4.13).

Comparison of the present case with case A1, where the collision is also symmetric when viewed in terms of null dust clouds, leads immediately to the question what, from the standpoint of waves, is the reason for two different types of fluid to appear in the region of interaction. The answer to this question is not hard to obtain, if one compares the values of φ on the $u=0$ and $v=0$ sides of region IV obtained using Eq. (4.16) with the ones resulting from Eq. (4.19). One then immediately realizes that in the first case the scalar waves meet in phase while in the second they have a phase difference of π rads when they meet.

Case A3—$\delta_1{}^2\equiv\delta^2=1/2,\delta_2=0(n=m=1)$: Restricting ourselves again to the physically significant quantities considered in the previous case, we find that

$$\varphi=2a\ln[1-u^2H(u)-v^2H(v)]+\delta\ln[(1-\eta)/(1+\eta)], \quad (4.22)$$

$$R_{ab}(u<0)=-4[(\delta+2av)/(1-v^2)]^2H(v)l_al_b, \quad (4.23a)$$

$$R_{ab}(v<0)=-4[(\delta+2au)/(1-u^2)]^2H(u)k_ak_b, \quad (4.23b)$$

$$R_{uv}=-\frac{4H(u)H(v)}{t^2\cos\chi\cos\omega}[a\sin 2\chi+\delta\cos(\chi-\omega)]\times[a\sin 2\omega+\delta\cos(\chi-\omega)]. \quad (4.23c)$$

As shown by the first two of Eqs. (4.23), the present subclass of solutions also represents the collision of two pulses of scalar waves that are free of impulsive components. The last of the above equations, however, shows that the superposition of the incoming waves after collision is such that one cannot regard the region of interaction as consisting of a fluid of a certain type. Specifically, suppose that $\delta a<0$, with $|\delta/2a|<1$. Then, it follows from Eq. (4.23c) that $R_{uv}=0$ along the following two curves of region IV. The first is defined by the relation $\sin 2\chi=-(\delta/a)\cos(\chi-\omega)$ and has its origin at the point $(u_0,0)$ with $u_0=|\delta/2a|$. Similarly, the second curve is defined by $\sin 2\omega=-(\delta/a)\cos(\chi-\omega)$ and starts at $(0,v_0)$, where $v_0=u_0$. Across these curves the sign of R_{uv} changes and, therefore, region IV is split into subregions occupied by a fluid whose type alternates among the varieties (i) and (ii) considered after Eq. (4.13) as illustrated in Fig. 1.

Case A4—$a=0,\delta_A=\delta\neq 0(1/n=2(1-4\delta^2),1/m=2)$: Along similar lines one finds that in this case

$$\varphi=\delta\ln[(1-\eta)(1-\mu)/(1+\eta)(1+\mu)] \quad (4.24)$$

and

$$R_{uu}=-16n^2\delta^2t^{-2}S^2u^{2n-2}H(u), \quad (4.25a)$$

$$R_{vv}=-[1-u^{2n}H(u)]^{-1}\delta(v)-4\delta^2t^{-2}S^{-2}u^{2n}H(u)H(v), \quad (4.25b)$$

$$R_{uv}=-e^{-M}\varepsilon=-8n\delta^2t^{-2}u^{2n-1}H(u)H(v). \quad (4.25c)$$

This is the case that best illustrates the advantage of the matter fields picture, as compared with the null currents one, in yielding a better understanding of the collision models. Specifically, from (4.25a) and (4.25b) it follows that the interior of region IV can be looked at as consisting of a pair of oppositely moving null dust currents. From (4.25c) on the other hand, it follows that the above currents exchange energy in a way that makes region IV appear to be occupied by an $\varepsilon=p$ perfect fluid.

Observe, however, that $\varepsilon\to 0$ as one approaches the $u=0$ hypersurface from the interior of region IV. Moreover, Eqs. (4.25a) and (4.25b) show that, while the radiation incident from the right in Fig. 1 has the form of a null dust cloud occupying the whole of the region III, only an impulsive shell of null dust riding the $v=0$ characteristic is incident from the left. This makes it hard to understand the origin of the right moving current appearing in region IV.

Consider, instead, the scalar field picture. In terms of the latter the present model represents the collision of a scalar plane wave incident from the right that at $(u,v)=(0,0)$ collides with an impulsive wave (shell of null dust). The focusing that results from this encounter is equivalent to the scalar wave pulse's entering a space-time region where the metric depends on both u and v. But, then, the Klein–Gordon equation, Eq. (2.5), implies that right moving waves are generated, i.e., backscattering occurs. This explains the appearance of the right moving currents in region IV, in the null current picture.

V. COLLISION OF ELECTROMAGNETIC AND NEUTRINO WAVES

Turning to examples of type B models that represent the collision of plane waves, let us assume that the functions A and B that specify Λ via Eq. (2.21) are such that

$$A(0) = B(0) = 0. \tag{5.1}$$

Let us further choose α and β such that

$$\alpha(0) = \beta(0) = 1/2, \tag{5.2}$$

and introduce the pair of functions (f,g) via

$$f(u) = \alpha(0) - \alpha(u), \quad g(v) = \beta(0) - \beta(v). \tag{5.3}$$

Then, Eq. (2.19) and the Khan–Penrose substitution (3.1) lead immediately to the following expressions for the metric coefficients and, thus, to a solution of Einstein's equations valid in all regions I–IV of Fig. 1:

$$e^{-M} = [1 - f(u)H(u) - g(v)H(v)]^{-1/2} \times e^{A(u)H(u) + B(v)H(v)}, \tag{5.4a}$$

$$e^{-U} = 1 - f(u)H(u) - g(v)H(v). \tag{5.4b}$$

Since $T_{ab} = -R_{ab}$ in the present case, Eq. (3.3) gives the following form for the stress-energy sources of the models under consideration:

$$T_{ab} = e^U[(f'' - f'A')H(u)k_a k_b + (g'' - g'B')H(v)l_a l_b + f'\delta(u)k_a k_b + g'\delta(v)l_a l_b]. \tag{5.5}$$

The only nonvanishing Weyl scalar, on the other hand, is given by

$$\Psi_2 = \tfrac{1}{6}e^M(M_{,uv} - U_{,uv}) = -\tfrac{1}{4}e^{M+2U}f'g'H(u)H(v). \tag{5.6}$$

The last expression implies that all of the models under consideration are conformally flat, unless $f'g' \neq 0$. We will show that the two nonvanishing, nonimpulsive terms in the expression (5.5) for T_{ab} can be interpreted as due to a pulse of electromagnetic and neutrino plane waves, respectively.

Consider in this respect the antisymmetric tensor field F_{ab} defined by

$$F_{ab} = \phi(l_a m_b - l_b m_a) + \bar{\phi}(l_a \bar{m}_b - l_b \bar{m}_a), \tag{5.7}$$

where $\phi = \phi(u,v)$, the covariant vector m_a is given by

$$m_a = (1/\sqrt{2})e^{-U/2}(\delta_a^2 + i\delta_a^3), \tag{5.8}$$

and the overbar denotes, now, complex conjugation. Provided F_{ab} satisfies Maxwell's equations

$$F^{ab}{}_{;b} = 0 = F_{[ab,c]}, \tag{5.9}$$

it represents a null electromagnetic field that gives rise to the stress-energy distribution

$$T^e_{ab} = F_{ac}F^c_{\ b} - \tfrac{1}{4}g_{ab}F_{cd}F^{dc} = 2\phi\bar{\phi}l_a l_b. \tag{5.10}$$

Comparing now Eqs. (5.5) and (5.10) we conclude that the second term in the right-hand side of the former can be attributed to an electromagnetic wave, provided one chooses a function ϕ such that

$$\phi\bar{\phi} = \tfrac{1}{2}e^U(g'' - g'B')H(v). \tag{5.11}$$

and,when the ϕ so chosen is substituted in Eq. (5.7), Eqs. (5.9) are satisfied.

Since there is no loss of generality by choosing ϕ to be real, we take

$$\phi = [\tfrac{1}{2}e^U(g'' - g'B')]^{1/2}H(v). \tag{5.12}$$

Substituting the last expression into Eq. (5.7), we find

$$F_{ab} = (g'' - g'B')^{1/2}H(v)(\delta_a^1\delta_b^2 - \delta_b^1\delta_a^2). \tag{5.13}$$

It is now an easy matter to verify that the last expression satisfies Eqs. (5.9) in all regions I–IV of Fig. 1, as well as on the separation hypersurfaces $u = 0$ and $v = 0$.

Let, on the other hand, φ_A stand for a two-spinor which in the spinor basis (O_A, ι_A) has the form

$$\varphi_A = e^{M/4}(\varphi o_A + \psi\iota_A), \tag{5.14}$$

where φ must not be confused with the scalar field considered in the last section. This spinor can be attributed to a neutrino field provided it satisfies Weyl's equations which in the present case read[12]

$$\varphi_{,v} = \tfrac{1}{2}U_{,v}\varphi, \tag{5.15a}$$

$$\psi_{,u} = \tfrac{1}{2}U_{,u}\psi. \tag{5.15b}$$

If we choose ψ to vanish, the corresponding stress-energy tensor becomes

$$T^n_{ab} = 2i(\varphi\bar{\varphi}_{,u} - \bar{\varphi}\varphi_{,u})k_a k_b. \tag{5.16}$$

Equivalently,

$$T^n_{ab} = 4e^U\rho^2\theta' k_a k_b, \tag{5.17}$$

since the solution of Eq. (5.15a) can be written in the form

$$\varphi = e^{U/2}\rho(u)e^{i\theta(u)}. \tag{5.18}$$

A particular choice of the functions ρ and θ appearing in the last expression is given by

$$\rho(u) = \tfrac{1}{2}(f'' - f'A')^{1/2}H(u), \tag{5.19a}$$

$$\theta(u) = u. \tag{5.19b}$$

When substituted into Eq. (5.17), it yields

$$T^n_{ab} = e^U(f'' - f'A')H(u)k_a k_b. \tag{5.20}$$

This expression can now be compared with the first term on the right-hand side of Eq. (5.5). It is hereby concluded that the latter term can be considered to be due to a neutrino plane wave which, according to Eqs. (5.14), (5.18), and (5.19) is given by

$$\varphi_A = \tfrac{1}{2}e^{(2U+M)/4}(f'' - f'A')^{1/2}e^{iu}H(u)o_A. \tag{5.21}$$

Having completed the proof that the extended type B solutions can be interpreted as representing, in general, the collision of a pulse of electromagnetic radiation incident from the left in Fig. 1 with a pulse of neutrino radiation incident from the right, let us turn to some specific examples.

Let us then choose the functions f and g to be of the form

$$f(u) = u^{2n}, \quad g(v) = v^{2m}, \tag{5.22}$$

where n and m satisfy conditions (4.2). Let the functions A and B, on the other hand, be given by

$$A = p\ln(1 - u^n) + q\ln(1 + u^n), \tag{5.23a}$$

$$B = r\ln(1 - v^m) + s\ln(1 + v^m), \tag{5.23b}$$

where the tertrad of constants (p,q,r,s) is such that

$$p=q\geqslant 0, \text{ when } n=1/2, \tag{5.24a}$$

$$r=s\geqslant 0, \text{ when } m=1/2, \tag{5.24b}$$

and

$$F(x)\equiv[n(p+q)-(2n-1)]x^{2n} + n(p-q)x^n+2n-1\geqslant 0, \tag{5.25a}$$

$$G(x)\equiv[m(r+s)-(2m-1)]x^{2m} + m(r-s)x^m+2m-1\geqslant 0, \tag{5.25b}$$

where $x\in[0,1)$, otherwise.

Equations (5.22) and (5.23) define a six-parameter family of solutions corresponding to stress-energy sources described by the tensor [cf. Eq. (5.5)].

$$T_{ab}=T^e_{ab}+T^n_{ab}+\frac{2mv^{2m-1}}{1-u^{2n}H(u)}\delta(v)l_al_b + \frac{2nu^{2n-1}}{1-v^{2m}H(v)}\delta(u)k_ak_b, \tag{5.26}$$

where

$$T^i_{ab}=\frac{2mv^{2m-2}}{(1-v^{2m})[1-u^{2n}H(u)-v^{2m}]}G(v)H(v)l_al_b \tag{5.27}$$

and

$$T^n_{ab}=\frac{2nu^{2n-2}}{(1-u^{2n})[1-u^{2n}-v^{2m}H(v)]} \times F(u)H(u)k_ak_b. \tag{5.28}$$

The nonimpulsive parts of the above sources can be attributed to a pulse of electromagnetic and neutrino waves given by

$$F_{ab}=2v^{m-1}\left[\frac{2mG(v)}{1-v^{2m}}\right]^{1/2}H(v)l_{[a}n_{b]}, \quad n_a=\delta^2_a \tag{5.29}$$

and

$$\varphi_A=\frac{u^{n-1}}{2}\left[\frac{2ne^{u+M/2}F(u)}{1-u^{2n}}\right]^{1/2}H(u)e^{iu}o_A, \tag{5.30}$$

respectively.

The above class of solutions covers a large variety of physically interesting subcases that are obtained by choosing the corresponding parameters appropriately. When the tetrad (p,q,r,s) vanishes and $n=m=1$, for example, we recover Griffiths'[12] solution representing the collision of a pair of constant profile shock waves made of photons and neutrinos, respectively. Keeping $n=m=1$ and varying $p,q,r,$ and s so that conditions (5.25) are satisfied, the profile of the above shock waves can be changed so as to become a function of v and u, respectively. For $p=q=0$, and $n=1/2, m\geqslant 1$ we obtain models representing the collision of an electromagnetic wave with an impulsive shell of null dust. The electromagnetic wave in the latter case can be replaced by a neutrino wave by choosing $r=s=0$ and $m=1/2, n\geqslant 1$.

Even though the detailed description of the physical behavior of any of the above particular examples is interesting in its own, we prefer to turn to the following observation which is related to the issues discussed in the Introduction. Consider in this direction the three-parameter subfamily of the solutions obtained from the above six-parameter family by setting

$$p=(2a+\delta_+)^2, \quad q=(2a-\delta_+)^2, \quad r=(2a+\delta_-)^2$$
$$s=(2a-\delta_-)^2, \quad n=1/2(1-\delta^2_+),$$
$$m=1/2(1-\delta^2_-), \tag{5.31}$$

where a,δ_A are the same constants that specify the family of scalar wave solutions analyzed in the last section. It is then easy to verify that, in the precollision regions I–III, the metric resulting from the choice (5.31) is identical to the one corresponding to the scalar wave models referred to above, even though the solutions differ from each other in region IV.

It is thereby shown that Taub's results on the ambiguity that characterizes planar colliding null dust clouds with impulsive leading fronts extend to solutions which are free from such impulsive components too. Namely, given the metric (solutions of Einstein's equations) in regions I–III, one would not be able to predict whether the solution in region IV will be of type A or type B. The collision process becomes determinate, however, once the stress-energy tensor or, equivalently, the type of interaction is preassigned in region IV. In our case, the resolution of the ambiguity was accomplished by constructing the corresponding stress-energy distributions, at least the nonimpulsive parts, from well-defined matter fields in the spirit of the arguments presented by Feinstein *et al.* in Ref. 5.

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